

# ALGEBRAIC VARIETIES WITH AUTOMORPHISM GROUPS OF MAXIMAL RANK

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ABSTRACT. We confirm, to some extent, the belief that a projective variety  $X$  has the largest number (relative to the dimension of  $X$ ) of independent commuting automorphisms of positive entropy only when  $X$  is birational to a complex torus or a quotient of a torus. We also include an addendum to an early paper [28] though it is not used in the present paper.

*Dedicated to the memory of Eckart Viehweg.*

## 1. INTRODUCTION

We work over the field  $\mathbb{C}$  of complex numbers. We consider an automorphism group  $G \leq \text{Aut}(X)$  of positive entropy on a compact complex Kähler manifold or normal projective variety  $X$ . Our belief is:  $X$  has the largest number (relative to the dimension of  $X$ ) of independent commuting automorphisms of positive entropy only when  $X$  is a complex torus or a quotient of a torus. We confirm this, to some extent, in Theorems 1.2 and 2.2. Our approach is conceptual and classification free. See [5] and [28] for the case of threefolds or minimal varieties.

For an automorphism  $g \in \text{Aut}(X)$ , its (topological) *entropy*  $h(g) = \log \rho(g)$  is defined as the logarithm of the *spectral radius*  $\rho(g)$  of its action on the cohomology:

$$\rho(g) := \max\{|\lambda|; \lambda \text{ is an eigenvalue of } g^*_{|\oplus_{i \geq 0} H^i(X, \mathbb{C})}\}.$$

By the fundamental work of Gromov and Yomdin, the above definition is equivalent to the original definition for automorphisms on compact Kähler manifolds or  $\mathbb{Q}$ -factorial projective varieties (cf. [10], and also [6, §2.2] and the references therein).

By the surface classification, a (smooth) compact complex surface  $S$  has some  $g \in \text{Aut}(S)$  of positive entropy only if  $S$  is either the projective plane blown up in at least 10 points, or obtained by blowing up some  $g$ -periodic orbits on a complex torus,  $K3$  surface or Enriques surface (cf. [3] for more details). See [26] for a similar phenomenon in higher dimension.

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In their very inspiring paper [6, Theorem 1], Dinh and Sibony have proved the following (cf. [25] for its generalization to solvable groups).

**Theorem 1.1.** (cf. [6, Theorem 1.1]) *Let  $X$  be a compact complex Kähler manifold of dimension  $n \geq 2$  and  $G \leq \text{Aut}(X)$  a commutative subgroup. Then  $G = N(G) \times G_1$  where  $N(G)$  consists of all elements in  $G$  of null entropy and is a subgroup of  $G$ , and  $G_1$  is a free abelian group of rank  $r = r(G_1) \leq n - 1$  and with  $g_1$  of positive entropy for all  $g_1 \in G_1 \setminus \{\text{id}\}$ . Further, if  $r = n - 1$  then  $N(G)$  is finite.*

There do exist examples of  $n$ -dimensional complex tori, Calabi-Yau varieties and rationally connected varieties  $X$  admitting *maximal* rank symmetries  $\mathbb{Z}^{\oplus n-1} \cong G \leq \text{Aut}(X)$  with every element in  $G \setminus \{\text{id}\}$  being of positive entropy; cf. [6, Example 4.5] and [28, Example 1.7]. All these examples are quotients of tori, as expected (cf. Theorem 1.2 below). Indeed, it is known that  $\text{SL}_n(\mathbb{Z})$  includes a free abelian group  $G$  of rank  $n - 1$  which has a natural faithful action on the complex  $n$ -torus  $X := E^n$  with  $g$  of positive entropy for every  $g \in G \setminus \{\text{id}\}$ , where  $E$  is an elliptic curve (cf. [6, Example 4.5] for details and references therein).

Our standing assumptions are now (i)  $G$  is commutative, (ii) every non-trivial element of  $G$  has positive entropy, and (iii)  $\text{rank } r(G) = \dim X - 1$ .

Theorem 1.2 below and Theorem 2.2 in §2 are our main results. In Theorem 2.2(3) a stricter restriction will be imposed on the  $Y$  of Theorem 1.2(2).

For the definitions of *Kodaira dimension*  $\kappa(X)$  and singularities of *terminal*, *canonical* or *klt type*, we refer to [15] or [16, Definitions 2.34 and 7.73]. A subvariety  $Y \subset X$  is called *G-periodic* if  $Y$  is stabilized (set theoretically) by a finite-index subgroup of  $G$ . Denote by  $q(X) := h^1(X, \mathcal{O}_X)$  the *irregularity* of  $X$ . A projective manifold  $Y$  is a *Q-torus* if  $Y = A/F$  for a finite group  $F$  acting freely on an abelian variety  $A$ .

**Theorem 1.2.** *Let  $X$  be an  $n$ -dimensional normal projective variety and let  $G := \mathbb{Z}^{\oplus n-1}$  act on  $X$  faithfully such that every element of  $G \setminus \{\text{id}\}$  is of positive entropy. Then the following hold:*

- (1) *Suppose that  $\tau : A \rightarrow X$  is a  $G$ -equivariant finite surjective morphism from an abelian variety  $A$ . Then  $\tau$  is étale outside a finite set (hence  $X$  has only quotient singularities and is klt);  $K_X \sim_{\mathbb{Q}} 0$  ( $\mathbb{Q}$ -linear equivalence); no positive-dimensional proper subvariety  $Y \subset X$  is  $G$ -periodic.*
- (2) *Conversely, suppose that no positive-dimensional  $G$ -periodic subvariety  $Y \subset X$  is either fixed (point wise) by a finite-index subgroup of  $G$ , or is a  $Q$ -torus with  $q(Y) > 0$ , or has  $\kappa(Y) = -\infty$ . Suppose also one of the following two conditions.*
  - (2a)  *$n = 3$ , and  $X$  is klt.*

(2b)  $n \geq 3$ , and  $X$  has only quotient singularities.

Then  $X \cong A/F$  for a finite group  $F$  acting freely outside a finite set of an abelian variety  $A$ . Further, for some finite-index subgroup  $G_1$  of  $G$ , the action of  $G_1$  on  $X$  lifts to an action of  $G_1$  on  $A$ .

As a consequence of Theorems 1.1 and 1.2, we have:

**Corollary 1.3.** *Let  $X$  be a normal projective variety of dimension  $n \geq 3$  with only quotient singularities, and let  $G := \mathbb{Z}^{\oplus r}$  act on  $X$  faithfully for some  $r \geq n - 1$  such that every element of  $G \setminus \{\text{id}\}$  is of positive entropy. Then  $r = n - 1$ . Further,  $X \cong A/F$  for a finite group  $F$  acting freely outside a finite set of an abelian variety  $A$ , if and only if  $X$  has no positive-dimensional  $G$ -periodic proper subvariety  $Y \subset X$ .*

The proof of Theorem 2.2 gives the following, which was also essentially proved in [28].

**Corollary 1.4.** *Let  $X$  be a normal projective variety of dimension  $n \geq 3$  and let  $G := \mathbb{Z}^{\oplus r}$  act on  $X$  faithfully for some  $r \geq n - 1$  such that every element of  $G \setminus \{\text{id}\}$  is of positive entropy. Then  $r = n - 1$ . Suppose further that both  $X$  and  $(X, G)$  are minimal in the sense of 2.1, and either  $X$  has only quotient singularities, or  $X$  is a klt threefold. Then  $X \cong A/F$  for a finite group  $F$  acting freely outside a finite set of an abelian variety  $A$ .*

**Remark 1.5.** (1) In Theorems 1.2 (2) and 2.2, we need to assume that  $\dim X \geq 3$  which is used at the last step 2.15 to show the vanishing of the second Chern class  $c_2(X)$ . In fact, inspired by the comment of the referee, one notices that a complete intersection  $X \subset \mathbb{P}^2 \times \mathbb{P}^2$  of two very general hypersurfaces of type  $(1, 1)$  and  $(2, 2)$  is a  $K3$  surface (called Wehler's surface) of Picard number two,  $X$  has an automorphism  $g$  of entropy  $2 \log(2 + \sqrt{3}) > 0$ , and  $X$  contains no  $(-2)$ -curve, so there is no  $g$ -periodic curve on  $X$ ; cf. [24, Theorem 2.5, Proposition 2.6], [22, Lemma 2.1, Proposition 2.5]. Thus, both  $X$  and the pair  $(X, \langle g \rangle)$  are minimal in the sense of 2.1. However,  $X$  is not birational to the quotient of a complex torus, because a (smooth) projective  $K3$  surface  $X$  birational to the quotient of a complex torus has the transcendental lattice of rank  $\leq$  (that of a complex 2-torus), i.e.,  $\leq 5$ , and hence has Picard number  $\geq (h^2(X, \mathbb{C}) - 5)$  which is 17.

(2) In Theorem 1.2 (2) (resp. Theorem 2.2), we can weaken the assumption on  $G$  as a condition on  $G^* := G|_{\text{NS}_{\mathbb{R}}(X)}$  (cf. [6, Theorem 4.7]):

“ $G^* \cong \mathbb{Z}^{\oplus n-1}$  (resp.  $G^* \cong \mathbb{Z}^{\oplus n-1}$ ,  $G$  is virtually solvable) and every element of  $G^* \setminus \{\text{id}\}$  is of positive entropy.”

But we need also to replace the last sentence

“Further, ... of  $G_1$  on  $A$ .” in Theorems 1.2 (2) and 2.2 as:

“Further, the action of  $G$  on  $X$  lifts to an action of a group  $\tilde{G}$  on  $A$  with  $\tilde{G}/\text{Gal}(A/X) \cong G$ .”

The virtual solvability of  $G$  is used in the middle of Claim 2.12 and end of the proof of Lemma 2.11.

Our bimeromorphic point of view, in terms of the minimality assumption in 1.4 and 2.2, towards the dynamics study seems natural, since one may blow up some Zariski-closed and  $G$ -stable proper subset of  $X$  (if such subset exists) to get another pair  $(X', G)$  which is essentially the same as the original pair  $(X, G)$ .

The very starting point of our proof is the existence of enough nef eigenvectors  $L_i$  of  $G$ , due to the fundamental work of Dinh-Sibony [6]. The minimal model program (cf. [16]) is used with references provided for non-experts. Our main contribution lies in Lemmas 2.10 and 2.11 where we show that the pair  $(X, G)$  can be replaced with an equivariant one so that  $H := \sum_{i=1}^n L_i$  is an ample divisor. To conclude, we prove a result of Hodge-Riemann type for singular varieties to show the vanishing of Chern classes  $c_i(X)$  ( $i = 1, 2$ ), utilizing  $H^{n-i} \cdot c_i(X) = 0 = H^{n-2} \cdot c_1(X)^2$ . Then we use the characterization of étale quotient of a complex torus as the compact Kähler manifold  $X$  with vanishing Chern classes  $c_i(X)$  ( $i = 1, 2$ ) (cf. [1, §1]) and its generalization to singular varieties (cf. [21]).

For a possible generalization of the proof to a Kähler  $n$ -fold  $X$ , we remark that the restriction ‘rank  $r(G) = n - 1$ ’ implies that  $X$  is either Moishezon and hence projective, or has algebraic dimension  $a(X) = 0$  (cf. [25, Theorem 1.2]). Thus the case  $a(X) = 0$  remains to be treated. See a related remark in [4, §3.6].

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## 2. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2 in the Introduction and its slightly generalized version Theorem 2.2 below.

We use the **terminology** and **notation** in [13], [15] and [16]. By  $G|_Y$ , we mean that there is a natural (from the context) induced action of  $G$  on  $Y$ .

**Definition 2.1.** A normal projective variety  $X$  is *minimal* if it has only *klt* singularities (cf. [16, Definition 2.34]) and the *canonical divisor*  $K_X$  is nef. Here a divisor  $D$  is *nef* if

the intersection  $D \cdot C = \deg(D|_C) \geq 0$  for every curve  $C$  in  $X$ . It is known that a quotient singularity is *klt* and the converse is true in dimension two (cf. [16, Propositions 5.20 and 4.18]).

Let  $G \leq \text{Aut}(X)$  be a subgroup. A pair  $(X, G)$  is *minimal* if: for every finite-index subgroup  $G_1 \leq G$ , every  $G_1$ -equivariant birational morphism  $\sigma : X \rightarrow X_1$  onto a variety  $X_1$  so that the pair  $(X_1, \Delta)$  is *klt* for some effective (boundary)  $\mathbb{R}$ -divisor  $\Delta$  (cf. [16, Definition 2.34]), is an isomorphism. Here  $\Delta$  is not required to be  $G_1$ -stable.

We remark that every fibre of such  $\sigma$  is rationally chain connected by [11, Corollary 1.5]. Thus every irreducible component of the *exceptional locus*  $\text{Exc}(\sigma) \subset X$  (the subset of  $X$  along which  $\sigma$  is not isomorphic) is  $G$ -periodic and uniruled.

The two minimality definitions above are slightly different from the ones in [28].

Although the Minimal Model Program predicts the existence of a minimal model  $X_m$  for every non-uniruled projective variety  $X$  and such existence is a theorem now for varieties of general type (cf. [2]), it is much harder to prove the existence of a minimal pair  $(X_m, G)$  for any given pair  $(X, G)$ , because the regular action of  $G$  on  $X$  induces a priori only a birational action of  $G$  on  $X_m$ .

The result below is more precise than 1.2(2) in terms of the restriction on  $Y$  in 2.2(3) and is applicable under the Good Minimal Model Program which predicts that 2.2 (4) and (5) are always true (cf. [2] for its recent breakthrough, Remark 2.3).

**Theorem 2.2.** *Let  $X$  be a normal projective variety of dimension  $n \geq 3$ , and let  $G := \mathbb{Z}^{\oplus r}$  act on  $X$  faithfully for some  $r \geq n - 1$  such that every element in  $G \setminus \{\text{id}\}$  is of positive entropy. Then  $r = n - 1$ . Suppose further the following five conditions.*

- (1) *Either  $X$  has only quotient singularities, or is a klt threefold.*
- (2) *The pair  $(X, G)$  is minimal in the sense of 2.1.*
- (3) *No positive-dimensional  $G$ -periodic subvariety  $Y \subset X$  is either fixed (point wise) by a finite-index subgroup of  $G$ , or a  $\mathbb{Q}$ -torus with  $q(Y) > 0$ , or a rational curve.*
- (4) *Every projective manifold  $Y$  with  $\dim Y \leq n - 1$ ,  $\kappa(Y) = -\infty$  and  $q(Y) > 0$ , is uniruled.*
- (5) *Every projective manifold  $Y$  with  $\dim Y \leq n - 2$ ,  $\kappa(Y) = 0$  and  $q(Y) > 0$ , has a good minimal model in the sense of Kawamata.*

*Then  $X \cong A/F$  for a finite group  $F$  acting freely outside a finite set of an abelian variety  $A$ . Further, for some finite-index subgroup  $G_1$  of  $G$ , the action of  $G_1$  on  $X$  lifts to an action of  $G_1$  on  $A$ .*

**Remark 2.3.** (1) The Condition (5) in Theorem 2.2 is always true when  $n = \dim X \leq 5$  (cf. [16, 3.13]).

- (2) The condition (4) in Theorem 2.2 is always true when  $n \leq 5$ , by applying Iitaka's  $C_{k,r}$  to the Albanese map  $\text{alb}_Y : Y \rightarrow \text{Alb}(Y)$ . Here the Albanese  $\text{Alb}(Y)$  of  $Y$  is a complex torus and every holomorphic map from  $Y$  to a complex torus factors through the albanese map  $\text{alb}_Y$  (cf. [23, Ch IV]). In particular, every subgroup  $H \leq \text{Aut}(Y)$  induces a natural action of  $H$  on  $\text{Alb}(Y)$  so that  $\text{alb}_Y$  is  $H$ -equivariant.

We begin with two lemmas.

**Lemma 2.4.** *Let  $G$  be a group and  $H \triangleleft G$  a finite normal subgroup such that*

$$G/H = \langle \overline{g_1} \rangle \times \cdots \times \langle \overline{g_r} \rangle \cong \mathbb{Z}^{\oplus r}$$

*for some  $r \geq 1$  and  $g_i \in G$ . Then there is an integer  $s > 0$  such that  $G_1 := \langle g_1^s, \dots, g_r^s \rangle$  satisfies*

$$G_1 = \langle g_1^s \rangle \times \cdots \times \langle g_r^s \rangle \cong \mathbb{Z}^{\oplus r}$$

*and it is a finite-index subgroup of  $G$ ; further, the quotient map  $\gamma : G \rightarrow G/H$  restricts to an isomorphism  $\gamma|_{G_1} : G_1 \rightarrow \gamma(G_1)$  onto a finite-index subgroup of  $G/H$ .*

*Proof.* We only need to find some  $s > 0$  such that  $g_i^s$  and  $g_j^s$  are commutative to each other for all  $i, j$ . Since  $G/H$  is abelian, the commutator subgroup  $[G, G] \leq H$ . Thus the commutators  $[g_1^t, g_2]$  ( $t > 0$ ) all belong to  $H$ . The finiteness of  $H$  implies that  $[g_1^{t_1}, g_2] = [g_1^{t_2}, g_2]$  for some  $t_2 > t_1$ , which implies that  $g_1^{s_{12}}$  commutes with  $g_2$ , where  $s_{12} := t_2 - t_1$ . Similarly, we can find integers  $s_{1j} > 0$  such that  $g_1^{s_{1j}}$  commutes with  $g_j$ . Set  $s_1 := s_{12} \times \cdots \times s_{1r}$ . Then  $g_1^{s_1}$  commutes with every  $g_j$ . Similarly, for each  $i$ , we can find an integer  $s_i > 0$  such that  $g_i^{s_i}$  commutes with  $g_j$  for all  $j$ . Now  $s := s_1 \times \cdots \times s_r$  will do the job. This proves the lemma.  $\square$

**Lemma 2.5.** *Let  $X$  be an  $n$ -dimensional projective variety,  $H_i$  nef and big  $\mathbb{R}$ -Cartier divisors and  $D$  an  $\mathbb{R}$ -Cartier divisor such that  $H_1 \cdots H_{n-1} \cdot D = 0$ . Then we have:*

- (1)  $H_1 \cdots H_{n-2} \cdot D^2 \leq 0$ .
- (2) *Suppose that  $H_1, \dots, H_{n-2}$  are ample divisors. Then  $H_1 \cdots H_{n-2} \cdot D^2 = 0$  holds if and only if  $D \equiv 0$  (numerically), i.e.,  $D \cdot C = 0$  for every curve  $C$  on  $X$ .*

*Proof.*  $H_1 \cdots H_{n-1} \cdot D = 0$  implies the assertion (1) by pulling back to a resolution of  $X$  (cf. [6, Corollary 3.4]).

We still need to prove the ‘only if’ part of the assertion (2) which will be done by induction on the dimension  $n$ . When  $n \leq 2$ , the assertion (2) follows from the Hodge index theory (for surfaces). Suppose that  $n \geq 3$  and

$$H_1 \cdots H_{n-1} \cdot D = H_1 \cdots H_{n-2} \cdot D^2 = 0.$$

Let  $\sigma : X' \rightarrow X$  be Hironaka's resolution such that  $-E$  is relatively ample for some  $\sigma$ -exceptional effective divisor  $E$ . Replacing  $E$  by its small multiple, we may assume that  $\sigma^*H_1 - E$  is ample (cf. [16, Proposition 1.45]).

For a curve  $C$  on  $X$ , we take an ample irreducible divisor  $A_1$  on  $X'$  such that  $\sigma(A_1)$  contains  $C$ . Take very small  $\varepsilon > 0$  such that  $\varepsilon A_1 \leq \sigma^*H_1 - E$ . Thus we can write  $\sigma^*H_1 - E \equiv \sum_{i=1}^s r_i A_i$  where  $r_i \in \mathbb{R}_{>0}$  and  $A_i$  are ample irreducible divisors. Since  $E$  is contracted by  $\sigma$  and by the projection formula, we have  $\sigma_*E = 0$  and

$$E \cdot \sigma^*D_1 \cdots \sigma^*D_{n-1} = \sigma_*E \cdot D_1 \cdots D_{n-1} = 0$$

for all Cartier divisors  $D_i$ . Set  $H'_i := \sigma^*H_i$ ,  $D' := \sigma^*D$ . Then

$$(1) \quad \sum_{i=1}^s r_i A_i \cdot \left( \prod_{j=2}^{n-1} H'_j \right) \cdot D' = (H'_1 - E) \cdot \left( \prod_{j=2}^{n-1} H'_j \right) \cdot D' = \left( \prod_{j=1}^{n-1} H'_j \right) \cdot D' = \left( \prod_{j=1}^{n-1} H_j \right) \cdot D = 0,$$

$$(2) \quad \sum_{i=1}^s r_i A_i \cdot \left( \prod_{j=2}^{n-2} H'_j \right) \cdot (D')^2 = (H'_1 - E) \cdot \left( \prod_{j=2}^{n-2} H'_j \right) \cdot (D')^2 = \left( \prod_{j=1}^{n-2} H'_j \right) \cdot (D')^2 = \left( \prod_{j=1}^{n-2} H_j \right) \cdot D^2 = 0.$$

By the equality (1) above and since  $A_i$  and  $H_j$  are nef, we have (for all  $i$ ):

$$(3) \quad (H'_{2|A_i}) \cdots (H'_{n-1|A_i}) \cdot (D'_{|A_i}) = A_i \cdot H'_2 \cdots H'_{n-1} \cdot D' = 0.$$

Hence  $A_i \cdot H'_2 \cdots H'_{n-2} \cdot (D')^2 \leq 0$  by the assertion (1). This together with the equality (2) above imply that for all  $i$ , we have

$$(4) \quad (H'_{2|A_i}) \cdots (H'_{n-2|A_i}) \cdot (D'_{|A_i})^2 = A_i \cdot H'_2 \cdots H'_{n-2} \cdot (D')^2 = 0.$$

Write  $B_i := \sigma(A_i)$  which is birational to  $A_i$ . By the equality (3) above,

$$\prod_{j=2}^{n-1} (H_{j|B_i}) \cdot (D_{|B_i}) = \prod_{j=2}^{n-1} ((\sigma^*H_j)_{|A_i}) \cdot ((\sigma^*D)_{|A_i}) = 0.$$

Similarly, the equality (4) above implies  $\prod_{j=2}^{n-2} (H_{j|B_i}) \cdot (D_{|B_i})^2 = 0$ . By the induction,  $D_{|B_i} \equiv 0$ . Note that  $B_1 = \sigma(A_1)$  contains  $C$ . Thus  $D \cdot C = (D_{|B_1}) \cdot C = 0$ . The lemma is proved.  $\square$

We now prove Theorems 1.2 and 2.2.

Let  $\tau : \tilde{X} \rightarrow X$  be a  $G$ -equivariant resolution due to Hironaka (cf. [7, (2.0)] and the reference therein). Applying the proof of [6, Theorems 4.7 and 4.3] to the action of  $G$  on the pullback  $\tau^* \text{Nef}(X)$  of the nef cone  $\text{Nef}(X)$  (instead of the Kähler cone of  $\tilde{X}$  there), we get nef  $\mathbb{R}$ -Cartier divisors  $\tau^*L_i$  ( $1 \leq i \leq n$ ) on  $\tilde{X}$  (resp.  $L_i$  on  $X$ ) as common

eigenvectors of  $G$  such that the intersection  $L_1 \cdots L_n \neq 0$  and the homomorphism below is an isomorphism onto a spanning lattice (where we write  $g^*L_i = \chi_i(g)L_i$ ):

$$\begin{aligned} \varphi : G &\rightarrow (\mathbb{R}^{n-1}, +) \\ g &\mapsto (\log \chi_1(g), \dots, \log \chi_{n-1}(g)). \end{aligned}$$

Since  $L_1 \cdots L_n = g^*(L_1 \cdots L_n) = \chi_1(g) \cdots \chi_n(g) L_1 \cdots L_n$ , we have

$$\chi_1 \cdots \chi_n = 1.$$

Set

$$H := \sum_{i=1}^n L_i.$$

Then  $H^n \geq L_1 \cdots L_n > 0$  and hence  $H$  is a nef and big  $\mathbb{R}$ -Cartier divisor.

**Lemma 2.6.** *Under the assumption of Theorem 1.2(2) or 2.2, the following are equivalent.*

- (1)  $H$  is not ample.
- (2)  $H^k \cdot Y = (H|_Y)^k = 0$  for some proper subvariety  $Y \subset X$  of dimension  $k > 0$ .
- (3)  $X$  has a  $G$ -periodic proper subvariety  $Y \subset X$  of dimension  $k > 0$ .

*Proof.* By Campana-Peternell's  $\mathbb{R}$ -divisor version of Nakai-Moishezon ampleness criterion, the assertions (1) and (2) are equivalent.

For (3)  $\Rightarrow$  (2), assume the assertion (3). After  $G$  is replaced by its finite-index subgroup  $G_1$  and noting that  $r(G_1) = r(G) = n - 1$ , we may assume that  $Y$  is stabilized by  $G$ . Note that for all  $i_j$ , we have

$$(5) \quad L_{i_1} \cdots L_{i_k} \cdot Y = 0.$$

Indeed, since  $\varphi(G) \subset \mathbb{R}^{n-1}$  is a spanning lattice,  $k \leq n - 1$ , and  $\chi_1 \cdots \chi_n = 1$ , we can choose  $g \in G$  such that  $\chi_{i_j}(g) > 1$  for all  $i_j$ . Acting on the left hand side of the equality (5) (a scalar) with  $g^*$  and noting that  $g^*Y = Y$ , we conclude the equality (5). This, in turn, implies that  $H^k \cdot Y = 0$ .

For (2)  $\Rightarrow$  (3), assume that  $H^k \cdot Y = 0$  as in the assertion (2). Then  $H|_Y$  is nef but not big. Write  $H = L + E$  with  $L$  ample and  $E$  effective (cf. the proof of [16, Proposition 2.61]). Since  $H|_Y$  is not big,  $Y \subseteq \text{Supp } E$ . Since  $L_{i_j}$  are all nef,  $H^k \cdot Y = 0$  means  $L_{i_1} \cdots L_{i_k} \cdot Y = 0$  for all  $i_j$ . Since  $L_{i_j}$  are all  $g^*$ -eigenvectors, reversing the process, we get  $H^k \cdot g(Y) = 0$  and hence  $g(Y) \subset \text{Supp } E$  by the above reasoning. The Zariski-closure  $\overline{\cup_{g \in G} g(Y)}$  is  $G$ -stabilized and contained in  $\text{Supp } E$ . Every irreducible component of this closure is a positive-dimensional  $G$ -periodic proper subvariety of  $X$ . This proves the assertion (3).  $\square$



2.7. In the proofs below, we will apply the Minimal Model Program to a pair  $(X, D)$  of a variety  $X$  and an effective  $\mathbb{R}$ -divisor  $D$  where the pair has at worst *klt* singularities (cf. [16, Definition 2.34]). If  $K_X + D$  is not nef, then there is a  $(K_X + D)$ -negative extremal ray  $R = \mathbb{R}_{>0}[\ell]$  of the *closed cone*  $\overline{\text{NE}}(X)$  of effective 1-cycles. Now the cone theorem [16, Theorem 3.7] gives rise to an extremal contraction  $\varphi : X \rightarrow Y$  to a normal variety  $Y$  such that a curve  $C$  is contracted by  $\varphi$  to a point if and only if the class  $[C]$  belongs to the extremal ray  $R$ . There are exactly three types of such  $\varphi$  (cf. [16, §3.7] for details):

- (i)  $\varphi$  is *divisorial*. It is a birational morphism whose exceptional locus  $\text{Exc}(\varphi) \subset X$  (the locus where  $\varphi$  is not isomorphic) is a prime divisor.
- (ii)  $\varphi$  is a *flip*. Then there is a *flipping*  $X \dashrightarrow X^+$  which is a rational map and isomorphic in codimension one. Further, there is a birational morphism  $X^+ \rightarrow Y$  such that the composite  $X \dashrightarrow X^+ \rightarrow Y$  coincides with  $\varphi$ . In particular, there is a natural isomorphism between the Neron-Severi groups (with  $\mathbb{R}$ -coefficient) of  $X$  and  $X^+$ . Such  $X^+$  is unique. Indeed,  $X^+ = \text{Proj } \bigoplus_{m \geq 0} \varphi_* \mathcal{O}_X(\lfloor m(K_X + D) \rfloor)$ .
- (iii)  $\dim Y < \dim X$ . Then  $\varphi$  is called a *Fano fibration* so that the restriction  $-(K_X + D)|_F$  of the anti-adjoint divisor to a general fibre  $F$  of  $\varphi$  is ample.

**Lemma 2.8.** *Under the assumption of Theorem 1.2(2) or Theorem 2.2,  $K_X + sH$  is nef and big for some (and hence all)  $s \gg 1$ .*

*Proof.* Write  $H = E/k + A_k$  with  $A_k$  a general ample  $\mathbb{Q}$ -divisor and  $E$  an effective  $\mathbb{R}$ -divisor (cf. the proof of [16, Proposition 2.61]). By the assumption,  $X$  is *klt*. Hence we can choose  $k \gg 1$  so that  $(X, E/k + A_k)$  is *klt* (cf. [16, Corollary 2.35(2)]), where  $A_k$  is replaced by  $(\sum_{i=1}^m D_i)/m$  with  $D_i$  general members of  $|mA_k|$  for some  $m \gg 1$ . Replace  $H$  by  $E/k + A_k$  for some large  $k$  and fix an ample  $\mathbb{Q}$ -divisor  $M$ , such that  $K_X + H + M$  is nef and *klt*.

We may assume that  $K_X + sH$  is not nef for any  $s > 0$ . We now consider  $K_X + H$ , but  $H$  may be replaced by  $sH$  for some  $s \gg 1$ . By the cone theorem (cf. [16, Theorem 3.7] or [2, Corollary 3.8.2]), there are only finitely many  $(K_X + H)$ -negative extremal rays  $\mathbb{R}_{>0}[\ell]$  in  $\overline{\text{NE}}(X)$ . Replacing  $H$  by a larger multiple, we may assume that all these  $\ell$  satisfy  $H \cdot \ell = 0$  (i.e.,  $L_i \cdot \ell = 0$  for all  $i$ ) and  $K_X \cdot \ell < 0$ . Since  $L_i \cdot g^{-1}(\ell) = \chi_i(g)L_i \cdot \ell = 0$  and hence  $H \cdot g^{-1}(\ell) = 0$ , and  $K_X \cdot g^{-1}(\ell) = g^*K_X \cdot \ell = K_X \cdot \ell < 0$ ,  $g^{-1}(\ell)$  (also an extremal curve) satisfies the same conditions as  $\ell$ . So these finitely many extremal rays  $\mathbb{R}_{>0}[\ell]$  are permuted, and hence stabilized by a finite-index subgroup  $G_1$  of  $G$ . This  $G_1$  will be used later on.

By [8, Theorem 1.1(6)] (which extends the result of Birkar), there are some  $1 \geq \lambda_0 > 0$  and extremal ray  $\mathbb{R}_{>0}[\ell_0]$  such that  $K_X + H + \lambda_0 M$  is nef,  $(K_X + H) \cdot \ell_0 < 0$  (and hence  $H \cdot \ell_0 = 0$  and  $K_X \cdot \ell_0 < 0$ ) and  $(K_X + H + \lambda_0 M) \cdot \ell_0 = 0$ .

Let  $\varphi_0 : X \rightarrow Y$  be the extremal contraction corresponding to the extremal ray  $\mathbb{R}_{>0}[\ell_0]$ , which is  $G_1$ -equivariant, where  $G_1$  is as mentioned earlier on. Thus every positive-dimensional irreducible component  $F_i$  of  $\text{Exc}(\varphi_0)$  is  $G_1$ -periodic and hence  $G$ -periodic.

Suppose that  $\varphi_0$  is a Fano fibration. If  $Y$  is not a point, this contradicts [25, Lemma 2.10] since  $r(G_1) = r(G) = \dim X - 1$  now. If  $Y$  is a point, then  $X$  is Fano of Picard number one and the class of  $-K_X$  is ample and preserved by  $G$ . Thus a finite-index subgroup of  $G$  is contained in  $\text{Aut}_0(X)$  by the result of Lieberman and Fujiki (cf. [17, Proposition 2.2], [7, Theorem 4.8]). Hence  $G$  is of null entropy, a contradiction.

Suppose that  $\varphi_0$  is birational (i.e., divisorial or a flip). Let  $H_Y$  and  $M_Y$  be the direct image on  $Y$  of  $H$  and  $M$ , respectively. Since  $K_X + H + \lambda_0 M$  is perpendicular to  $\ell_0$ , it is the pullback of  $K_Y + H_Y + \lambda_0 M_Y$  (cf. [16, Theorem 3.7(4)] or [2, Corollary 3.9.1]), so the latter adjoint divisor on  $Y$  (or the pair  $(Y, H_Y + \lambda_0 M_Y)$ ) is *klt* because so is its birational pullback on  $X$ . Hence for every irreducible component  $F_i$  of  $\text{Exc}(\varphi_0)$ , the fibres of  $\varphi_{0|F_i} : F_i \rightarrow \varphi_0(F_i)$  are all rationally chain connected by [11, Corollary 1.5]. Thus  $F_i$  are all  $G$ -periodic and uniruled (and hence of Kodaira dimension  $-\infty$ ). This contradicts the assumption of Theorem 1.2 (2), and Theorem 2.2 (2) as well.

Thus  $K_X + sH$  is nef for some  $s > 0$  and the lemma follows since  $H$  is big (and nef).  $\square$

**Lemma 2.9.** *Under the assumption of Theorem 1.2(2) or Theorem 2.2,  $K_X + sH$  is ample for some (and hence all)  $s \gg 1$ . Moreover,  $H|_Y \equiv 0$  (numerically) for every  $G$ -periodic subvariety  $Y \subset X$ , and hence  $(K_X)|_Y$  is ample.*

*Proof.* By Lemma 2.8, we may assume that  $K_X + H$  is nef and big and *klt* after replacing  $H$  by its large multiple. By the effective base-point freeness theorem (cf. [2, Theorem 3.9.1]), there is a birational morphism  $\psi : X \rightarrow Z$  onto a normal projective variety  $Z$ , such that  $K_X + H = \psi^*P$  for some ample divisor  $P \subset Z$ . Write  $H = E/k + A_k$  as in Lemma 2.8. Thus every extremal ray  $\mathbb{R}_{\geq 0}[\ell] \subset \overline{\text{NE}}(X)$  contracted by  $\psi$  is  $(K_X + E/k)$ -negative. By the cone theorem, there are only finitely many such extremal rays. Replacing  $H$  by its large multiple, we may assume that such  $\ell$  satisfies  $\ell \cdot H = 0 = \ell \cdot K_X$ , the condition of which is preserved by  $G$ . Thus we may assume that all such extremal rays are stabilized (resp. permuted) by a finite-index subgroup  $G_1$  of  $G$  (resp. by  $G$ ). In particular,  $\psi$  is  $G_1$ -equivariant;  $\text{Exc}(\psi)$  and every positive-dimensional irreducible component  $F_i$  of it is  $G$ -periodic. Let  $H_Z \subset Z$  be the direct image of  $H$ . Then  $K_X + H$  is the pullback of  $K_Z + H_Z$  ( $\sim_{\mathbb{Q}} P$ ) and hence the latter adjoint divisor on  $Z$  (or the pair  $(Z, H_Z)$ ) is *klt* because so is its pullback on  $X$ . As in Lemma 2.8, each  $F_i$  is  $G$ -periodic and uniruled, contradicting the assumption of Theorem 1.2 (2), and Theorem 2.2 (2) as well, unless  $\psi : X \rightarrow Z$  is an isomorphism, i.e.,  $K_X + H$  is ample.

For the second assertion, we claim the following vanishing of the intersection:

$$(6) \quad ((K_X + sH)|_Y)^{k-1} \cdot H|_Y = (K_X + sH)^{k-1} \cdot H \cdot Y = 0$$

where  $k = \dim Y$ . Indeed, since  $H = \sum_{i=1}^n L_i$ , the above intersection number is the summation of the following terms

$$K_X^{k-1-t} \cdot L_{j_1} \cdots L_{j_t} \cdot L_i \cdot Y$$

where  $0 \leq t \leq k-1 \leq n-2$ . Now the vanishing of each term above can be verified as in Lemma 2.6, since  $g^*K_X \sim K_X$  for  $g \in G$ . The equality (6) above is proved.

The equality (6) and ampleness of  $K_X + sH$  imply that the scalar

$$((K_X + sH)|_Y)^{k-2} \cdot (H|_Y)^2$$

is non-positive by Lemma 2.5, and hence is zero since  $K_X + sH$  and  $H$  are nef. Thus  $H|_Y \equiv 0$  by Lemma 2.5.  $\square$

**Lemma 2.10.** *Under the assumption of Theorem 1.2(2),  $H$  is ample.*

*Proof.* Suppose the contrary that  $H$  is not ample. Then by Lemma 2.6, a subvariety  $Y \subset X$  of positive-dimension  $k$  is  $G$ -periodic. We choose such  $Y$  with  $k \in \{1, \dots, n-1\}$  minimal. This  $Y$  is stabilized by a finite-index subgroup  $G_1$  of  $G$ . Now the class of the ample divisor  $(K_X)|_Y$  (cf. Lemma 2.9) is fixed by the pullback of every  $g \in G_1$ . Let  $\tau : Y' \rightarrow Y$  be a  $G_1$ -equivariant desingularization. Then  $\tau^*((K_X)|_Y)$  is nef and big on  $Y'$  and its class is fixed by the pullback of every  $g \in G_1$ . Thus  $G_{1|Y'} \leq \text{Aut}_0(Y')$  after  $G_1$  is replaced by a smaller finite-index subgroup of  $G$  by the result of Lieberman and Fujiki (cf. [26, Lemma 2.23], [17, Proposition 2.2], [7, Theorem 4.8]).

Suppose that the Kodaira dimension  $(\kappa(Y) :=) \kappa(Y') \geq 1$ . Then  $G$  (replaced by its finite-index subgroup) acts trivially on the base of the Iitaka fibration  $Y' \dashrightarrow B$  (with  $\kappa(Y') = \dim B$ ), by a classical result of Deligne-Nakamura-Ueno [23, Theorem 14.10]. Hence  $G$  stabilizes a general fibre  $Y'_b$  over a point  $b \in B$ . By the minimality of  $\dim Y = k > 0$ , we have  $\dim Y'_b = 0$  and hence  $Y'$  is of general type so that  $\text{Aut}(Y')$  (and hence  $G_{|Y'}$ ) are known to be finite; thus  $Y$  is fixed (point wise) by a subgroup of  $G|_Y \leq \text{Aut}(Y)$ , contradicting the assumption of Theorem 1.2(2).

Therefore, we may assume that  $\kappa(Y) \leq 0$ . Thus  $\kappa(Y) = 0$  by the assumption in Theorem 1.2(2), which is stronger than the condition (3) in Theorem 2.2.

For  $\text{Aut}_0(Y')$ , we shall use the terminology and results in [17, Theorem 3.12] or [7, §2, Theorem 5.5]. If  $\text{Aut}_0(Y')$  has a positive-dimensional linear part, then  $Y$  is ruled (which contradicts  $\kappa(Y) = 0$ ) by a classical result of Matsumura or its generalization [7, Proposition 5.10]; thus we may assume that the linear part of  $\text{Aut}_0(Y')$  is trivial; then the classical Jacobi homomorphism  $\text{Aut}_0(Y') \rightarrow \text{Aut}_0(\text{Alb}(Y')) \cong \text{Alb}(Y')$  has a finite

kernel (cf. [17, Theorem 3.12], [7, Theorem 5.5]). If  $(\dim \operatorname{Alb}(Y') =) q(Y') = 0$ , then  $G_{1|Y'} \leq \operatorname{Aut}_0(Y') = (1)$  and  $Y$  is fixed (point wise) by  $G_1$ , contradicting the assumption of Theorem 1.2(2).

Thus we may assume that  $q(Y') > 0$ . The singular locus  $\operatorname{Sing} Y$  is clearly stabilized by  $G_1$ . By the minimality of  $\dim Y = k > 0$ ,  $\operatorname{Sing} Y$  is empty or finite. Since  $\kappa(Y) = 0$ ,  $K_{Y'} \sim_{\mathbb{Q}} D$  for some effective  $\mathbb{Q}$ -divisor  $D$ . Since  $g^*D \sim_{\mathbb{Q}} g^*K_{Y'} \sim K_{Y'} \sim_{\mathbb{Q}} D$  for  $g \in G_1$ , we have  $g^*D = D$  for  $\kappa(Y') = 0$ . Thus all irreducible components of  $D \subset Y'$  and their images on  $Y$  are  $G_1$ -periodic. The minimality of  $\dim Y = k > 0$  implies that  $D$  is contracted to a few points on  $Y$  and also on  $Y^n$ , if we factor  $Y' \rightarrow Y$  as  $Y' \rightarrow Y^n \rightarrow Y$  with  $Y^n \rightarrow Y$  the normalization. Thus  $K_{Y^n} \sim_{\mathbb{Q}} 0$ , since it is the direct image of  $K_{Y'}$  ( $\sim_{\mathbb{Q}} D$ ). Further,  $Y^n$  has at worst canonical singularities since  $K_{Y'}$  is the pullback of  $K_{Y^n}$  plus an effective divisor  $D$ .

By the proof of [19, Theorem B] (cf. also [1, §3, especially Proposition 3]), there is a finite étale morphism  $F \times A \rightarrow Y^n$  such that  $A$  is an abelian variety,  $F$  is a weak Calabi-Yau variety (and hence  $q(F) = 0$ ) and

$$M := \operatorname{Aut}_0(Y^n)$$

lifts to a *split* action of  $\widetilde{M}$ , with  $\widetilde{M}/(\operatorname{Gal}((F \times A)/Y^n)) \cong M$ , on the product  $F \times A$ . Since  $q(F) = 0$  and hence  $\operatorname{Aut}_0(F) = (1)$  (cf. [19, Lemma 4.4]), the connected algebraic group  $\widetilde{M}$  acts on the factor  $F$  trivially. Thus  $\widetilde{M}$  stabilizes all fibres  $\{f\} \times A$ , and hence  $G_{1|Y^n}$  and  $G_{1|Y}$  stabilize their images (i.e., quotients of tori  $\{f\} \times A$ ), which form a so called torus-quotient covering family of  $Y^n$ .

By the minimality of  $\dim Y = k > 0$ , we have  $\dim Y = \dim A (\geq q(Y') > 0)$ , i.e.,  $\dim F = 0$ . Thus  $Y^n$  is a  $Q$ -torus (by taking the Galois closure of  $A \rightarrow Y^n$ ). In particular,  $Y^n$  is smooth and we may take  $Y' = Y^n$ .

By the assumption in Theorem 1.2(2),  $Y$  is not a  $Q$ -torus with  $q(Y) > 0$ . Hence  $Y^n \neq Y$ . Namely,  $\operatorname{Sing} Y$  is a non-empty finite set. Its inverse on  $Y^n$  is stabilized by  $G_{1|Y'}$ . Further, the image on  $\operatorname{Alb}(Y^n)$  of this (finite) inverse is stabilized by the image of  $G_{1|Y^n}$  in  $\operatorname{Aut}_0(\operatorname{Alb}(Y^n))$  (= translations) under the above Jacobi homomorphism. Hence such homomorphic image is trivial. So  $G_{1|Y^n}$  is a finite group because the Jacobi homomorphism has a finite kernel as mentioned earlier on. Thus  $Y^n$  and hence  $Y$  are fixed (point wise) by a finite-index subgroup of  $G$ . This contradicts the assumption in Theorem 1.2(2).  $\square$

**Lemma 2.11.** *Under the assumption of Theorem 2.2,  $H$  is ample.*

*Proof.* We use the argument and notation of Lemma 2.10. The condition (3) in Theorem 2.2 and the argument in Lemma 2.10 imply that  $\kappa(Y) := \kappa(Y') = -\infty$ . We may assume

that  $G_{1|Y'}$  is an infinite group, otherwise,  $Y'$  and hence  $Y$  are fixed (point wise) by a finite-index subgroup of  $G$ , contradicting the condition (3) in Theorem 2.2.

**Claim 2.12.** *The irregularity  $q(Y') > 0$ .*

We prove the claim. Suppose the contrary that  $q(Y') = 0$ . Then  $\text{Aut}_0(Y')$  is an affine algebraic group. Let  $\overline{G}_1 \leq \text{Aut}_0(Y')$  be the identity connected component of the closure of  $G_{1|Y'} \leq \text{Aut}_0(Y')$  in the Zariski topology which is abelian because so is  $G_1$ . By the minimality of  $\dim Y = k > 0$ , the induced action of the affine algebraic group  $\overline{G}_1$  on the normalization  $Y^n$  of  $Y$  has a Zariski-dense open orbit  $\overline{G}_1 y$ , a few isolated orbits and no others. Since  $\overline{G}_1$  is abelian, it is solvable and has a fixed point (by Borel's fixed point theorem), so it does have an isolated orbit in  $Y^n$ . By [14, Theorem and its Remark, p. 182],  $Y^n$  is a projective cone over a rational homogeneous projective manifold which again has a fixed point by the induced action of  $\overline{G}_1$ . Thus at least one of the lines generating the cone  $Y^n$  is stabilized by  $G_1$ ; the image of this line in  $Y$  is a  $G$ -periodic rational curve, contradicting the condition (3) in Theorem 2.2. This proves the claim.

By Claim 2.12,  $q(Y') > 0$  and hence  $Y'$  is not rationally connected (cf. [19, Remark 5.1]). Then  $Y'$  is uniruled, by the condition (4) of Theorem 2.2 and since  $\kappa(Y') = -\infty$ .

Note that the class of the ample divisor  $(K_X)_{|Y}$  is preserved by  $g^*$  ( $g \in G_1 \leq G$ ). We now apply results in [18, Theorem 4.18, Corollary 4.20] (cf. also [20, Lemma 4.1]). There is a 'special maximal rationally connected fibration'  $Y \dashrightarrow Z$  onto a normal projective variety  $Z$  such that its graph  $W = \Gamma_{Y/Z}$  is equi-dimensional over  $Z$  (i.e., every fibre  $W_z$  over a point  $z \in Z$  is of pure-dimension equal to  $\dim W - \dim Z$ ) and  $G_{1|Y}$  descends to  $G_{1|Z}$  with  $g^* H_Z \sim H_Z$  ( $g \in G_1$ ) for an ample divisor  $H_Z$  (the intersection sheaf of  $(K_X)_{|Y}$  over  $Z$ ). In particular, the natural maps  $W \rightarrow Y$  and  $W \rightarrow Z$  are all  $G_1$ -equivariant.

Let  $Z' \rightarrow Z$  be a  $G_1$ -equivariant resolution. Set  $Z' = Z$  when  $Z$  is smooth. Then  $q(Z') = q(Y') > 0$  (cf. [19, Lemma 5.3]). Since  $G_1$  acts trivially on the class of the nef and big pullback on  $Z'$  of  $H_Z$ , we may assume that  $G_{1|Z'} \leq \text{Aut}_0(Z')$  after replacing  $G_1$  by another finite-index subgroup of  $G$ , by the result of Lieberman and Fujiki (cf. [26, Lemma 2.23], [17, Proposition 2.2], [7, Theorem 4.8]).

Since  $Y'$  is not rationally connected as mentioned above, we have  $1 \leq \dim Z \leq \dim Y - 1 = k - 1 \leq n - 2$ . By [9, Corollary 1.4],  $Z$  is not uniruled. Thus  $\kappa(Z') \geq 0$ , by the condition (4) in Theorem 2.2 and since  $q(Z') > 0$ .

As in Lemma 2.10, we may assume that  $\kappa(Z') = 0$ , by the minimality of  $\dim Y = k > 0$ . By the condition (5) of Theorem 2.2,  $Z'$  has a good minimal model  $Z_m$  i.e.,  $Z_m$  has only canonical singularities and  $K_{Z_m} \sim_{\mathbb{Q}} 0$ . Take a partial resolution  $\tau : Z_m' \rightarrow Z_m$  such that

$Z'_m$  has only terminal singularities and  $K_{Z'_m} = \tau^* K_{Z_m} (\sim_{\mathbb{Q}} 0)$  (cf. [2, Corollary 1.4.3]). Replacing  $Z_m$  by  $Z'_m$  we may assume that  $Z_m$  has only terminal singularities.

We may assume that  $G_1|_{Z'}$  and hence

$$M := \text{Aut}_0(Z')$$

are non-trivial. The (non-trivial) birational action of the connected algebraic group  $M$  on the terminal minimal variety  $Z_m$  is biregular (cf. [12, Corollary 3.8]).

By the argument for  $Y^n$  in Lemma 2.10, either  $M$  stabilizes every member of a covering torus-quotient family on  $Z_m$  so that the inverse on  $Y$  of a general member is  $G$ -periodic which contradicts the minimality of  $\dim Y = k > 0$ , or  $Z_m$  is a  $Q$ -torus with  $q(Z_m) = q(Z') > 0$  and an étale covering  $B \rightarrow Z_m$  from an abelian variety  $B$  such that the action of  $M$  on  $Z_m$  lifts to an action of  $\widetilde{M}$  on  $B$  with  $\widetilde{M}/\text{Gal}(B/Z_m) \cong M$ .

We only need to consider this latter case.

**Claim 2.13.** *We have  $Z = Z' = Z_m$ .*

We prove this claim. Since  $Z_m$  is a  $Q$ -torus, it has no rational curves, and hence the birational map  $Z' \dashrightarrow Z_m$  is actually holomorphic (and  $M$ -equivariant). If  $Z' \rightarrow Z_m$  is not an isomorphism, then its non-isomorphic points on  $Z_m$  and its inverse on  $B$  form Zariski-closed subsets of codimension  $\geq 1$  which are respectively stabilized by the connected algebraic groups  $M|_{Z_m}$  and  $\widetilde{M}|_B \leq \text{Aut}_0(B)$  (= translations). By [27, Lemma 2.11], there is an  $\widetilde{M}$ -equivariant quotient map  $B \rightarrow B/C$  (with  $C$  a subtorus of dimension in  $\{0, \dots, \dim B - 1\}$ ) such that every element of  $\widetilde{M}|_{(B/C)} \leq \text{Aut}_0(B/C)$  (= translations) has a periodic point. Thus  $\widetilde{M}|_{(B/C)}$  is trivial. So a general coset of  $B/C$  (or a general curve on  $B$ , when  $\dim C = 0$ ), its image in  $Z_m$ , and the inverse of this image on  $Y$  are respectively stabilized by  $\widetilde{M}|_B$ ,  $M|_{Z_m}$  and  $G_1$ , contradicting the minimality of  $\dim Y = k > 0$ .

So we may assume that  $Z' = Z_m$ . If  $Z' \rightarrow Z$  is not an isomorphism, then the inverse on  $B$  of its non-isomorphic points on  $Z_m$  form a Zariski-closed subset of codimension  $\geq 1$  which is stabilized by  $\widetilde{M}|_B$ . This contradicts the minimality of  $\dim Y = k > 0$  by the same argument above. The claim is proved.

By Claim 2.13, we have  $Z = Z' = Z_m$ . If  $W = \Gamma_{Y/Z} \rightarrow Y$  is not isomorphic then its non-isomorphic points on  $W$  maps to a  $G_1$ -stabilized Zariski-closed subset of  $Z$  of codimension  $\geq 1$ , since  $W \rightarrow Z$  is equi-dimensional and  $G_1$ -equivariant. This leads to a contradiction as in the proof of Claim 2.13 above.

Hence we may assume that  $W = Y$ . By the same reasoning, we can show that  $Y$  is smooth and  $W = Y \rightarrow Z = Z_m$  is smooth.

Let  $\overline{G}_1 \leq \text{Aut}_0(Y)$  be the identity connected component of the closure of  $G_1|_Y \leq \text{Aut}_0(Y)$  in the Zariski topology, which is abelian because so is  $G_1$ . By the minimality

of  $\dim Y = k > 0$ , our  $Y$  has a dense open orbit  $\overline{G}_1 y$  intersecting every fibre of  $Y \rightarrow Z$ , and its complement  $\Sigma$  in  $Y$  is a finite set. In fact,  $\Sigma = \emptyset$ . Otherwise, the union of fibres of  $Y \rightarrow Z$  passing through the points in  $\Sigma$  is  $\overline{G}_1$ -stabilized and hence  $G$ -periodic, contradicting the minimality of  $k = \dim Y$ . Thus,  $Y$  is  $\overline{G}_1$ -homogeneous and  $Y \cong \overline{G}_1/F$  for a subgroup  $F \leq \overline{G}_1$ .

If  $\overline{G}_1$  is a complex torus, then so is  $Y$ , a contradiction to the condition (3) in Theorem 2.2. Therefore, the linear part  $L(\overline{G}_1)$  of  $\overline{G}_1$  is positive-dimensional and a rational variety by a result of Chevalley. Then every  $L(\overline{G}_1)$ -orbit on  $Y$  is a unirational variety and hence contained in a fibre of  $Y \rightarrow Z$  (since the  $Q$ -torus  $Z$  contains no rational curves). Thus the connected linear group  $L(\overline{G}_1)$  acts trivially on the base  $Z$  of the fibration  $Y \rightarrow Z$ . Since  $L(\overline{G}_1)$  is abelian (and hence solvable), its fixed locus (point wise) on each fibre of  $Y \rightarrow Z$  is non-empty and a proper subset of the fibre. So the union of these fixed loci on fibres is  $\overline{G}_1$ -stabilized (and a proper subset of  $Y$ ) because  $L(\overline{G}_1) \triangleleft \overline{G}_1$ . Thus  $Y$  is not  $\overline{G}_1$ -homogeneous, a contradiction. Lemma 2.11 is proved.  $\square$

**Lemma 2.14.** *Under the assumption of Theorem 1.2(2) or Theorem 2.2, the canonical divisor satisfies:  $K_X \equiv 0$  (numerically).*

*Proof.* By Lemmas 2.10 and 2.11,  $H$  is ample. Since  $g^*K_X \sim K_X$  for  $g \in G$ , by the proof of Lemma 2.6, we have  $H^{n-1} \cdot K_X = 0 = H^{n-2} \cdot K_X^2$ . Then  $K_X \equiv 0$  by Lemma 2.5.  $\square$

## 2.15. Proof of Theorems 1.2(2) and 2.2

By Theorem 1.1,  $r = n - 1$ . By Lemma 2.14, we have  $K_X \equiv 0$ . Let  $m > 0$  be minimal such that  $mK_X \sim 0$  (a result of Kawamata). Replacing  $X$  by its global index-one cover

$$\text{Spec } \oplus_{i=0}^{m-1} \mathcal{O}_X(-iK_X)$$

which has a natural  $G$ -action and whose canonical divisor is linearly equivalent to zero, we may assume that  $K_X (\sim 0)$  is Cartier and hence  $X$  has at worst Gorenstein canonical singularities. Let  $\sigma : X' \rightarrow X$  be a  $G$ -equivariant resolution crepant in codimension two. Denote by  $c_i(X')$  the  $i$ -th Chern class of  $X'$ . As in [21, p. 265] (cf. also [28, Definition 2.4]), define the second Chern class of  $X$  as  $c_2(X) := \sigma_* c_2(X')$  and regard it as a multi-linear form on  $\text{NS}_{\mathbb{C}}(X) \times \cdots \times \text{NS}_{\mathbb{C}}(X)$ :

$$c_2(X) \cdot D_1 \cdots D_{n-2} := \sigma_* c_2(X') \cdot D_1 \cdots D_{n-2} = c_2(X') \cdot \sigma^* D_1 \cdots \sigma^* D_{n-2}.$$

Since  $g^*c_2(X) = c_2(X)$  for all  $g \in G$ , we have the vanishing  $c_2(X) \cdot H^{n-2} = 0$  (using  $n \geq 3$  here) as in the proof of Lemma 2.6. Since  $H$  is ample by Lemmas 2.10 and 2.11, this vanishing and Miyaoka's pseudo-effectivity of  $c_2$  for minimal variety (cf. [21, Theorem 4.1, Proposition 1.1]) imply that  $c_2(X) = 0$  as a multi-linear form, as remarked in [28,

Definition 2.4]. Now the vanishing of  $c_i(X)$  ( $i = 1, 2$ ) imply that there is a finite surjective morphism  $A' \rightarrow X$  from an abelian variety  $A'$  (cf. [21, Corollary, p. 266] and [5, Theorem 7.6]). Indeed, when  $X$  has only quotient singularities, the vanishing of  $c_2(X)$  implies the vanishing of the orbifold second Chern class of  $X$  (cf. [21, Proposition 1.1]). Since  $K_{A'} \sim 0 \equiv K_X$ , the map  $A' \rightarrow X$  is étale in codimension one. Let  $A \rightarrow X$  be the Galois cover corresponding to the unique maximal lattice  $L$  in  $\pi_1(X \setminus \text{Sing } X)$  so that  $A$  is an abelian variety. Then  $X = A/F$  with  $F = \pi_1(X \setminus \text{Sing } X)/L = \text{Gal}(A/X)$ , and there is an exact sequence

$$1 \rightarrow F \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

where  $\tilde{G}$  (the lifting of  $G$ ) is a group acting faithfully on  $A$  (cf. the proof in [1, §3, especially Proposition 3] applied to étale-in-codimension-one covers, also [20, Proposition 3.5]). This proves Theorems 1.2(2) and 2.2; indeed, the last part is the application of Lemma 2.4 to the groups  $F \triangleleft \tilde{G}$ ; see the proof of Theorem 1.2(1) below for the freeness of the action  $F$  outside a finite set.

### 2.16. Proof of Theorem 1.2(1)

Suppose that a positive-dimensional proper subvariety  $Y' \subset X$  is  $G$ -periodic. Then a proper subvariety  $Y \subset A$  (dominating  $Y'$ ) is  $G$ -periodic and hence stabilized by a finite-index subgroup  $G_1 \leq G$ . By the proof of [27, Lemma 2.11], there is a  $G_1$ -equivariant homomorphism  $A \rightarrow A/B$  with  $\dim(A/B) \in \{1, \dots, n-1\}$ . This contradicts [25, Lemma 2.10], because  $r(G_{1|A}) = r(G_{1|X}) = \dim A - 1$  now (cf. [19, Appendix, Lemma A.8]). Hence the last assertion is true.

Let  $Y \subset A$  be the subset where  $\tau : A \rightarrow X$  is not étale. Then  $Y$  is  $G$ -stabilized. Hence  $\dim Y = 0$  by the argument above. Thus  $K_A = \tau^* K_X$  by the ramification divisor formula. Hence  $0 \sim \tau_* K_A = \tau_* \tau^* K_X = d K_X$  with  $d = \deg(\tau)$ . Replacing  $A$  by the Galois closure of  $\tau$ , we may assume that  $\tau : A \rightarrow X$  is Galois and hence  $X$  has only quotient singularities. This proves Theorem 1.2(1).

### 2.17. Proof of Corollary 1.3

By Theorem 1.1,  $r = n - 1$ . The ‘if’ part follows from Theorem 1.2(2). For the ‘only if’ part, the proof of Theorem 2.2 in 2.15 (using Lemma 2.4) implies the lifting of a finite-index subgroup  $G_1 \leq G$  to some complex torus cover of  $X$ . Hence  $X$  has no  $G_1$ -periodic (i.e.,  $G$ -periodic) subvariety  $Y \subset X$  of positive-dimension by Theorem 1.2(1). This proves the corollary.

### 2.18. Proof of Corollary 1.4



By Theorem 1.1,  $r = n - 1$ . We have  $H^{n-1}.K_X = 0$ , by the proof of Lemma 2.6 and since  $g^*K_X \sim K_X$  for  $g \in G$ . Thus  $K_X \equiv 0$ , by [20, Lemma 2.2] and since  $K_X$  is nef by the minimality of  $X$ . Hence  $H$  is ample by the proof of Lemma 2.9. The rest follows from the proof of Theorem 2.2 in 2.15.

### 2.19. Addendum to [28]

[28] itself is not used in the present paper. Thanks to the careful reading of the referee of the present paper, in the statements of [28, Theorem 1.1(2)] and [28, Corollary 1.2] (resp. [28, Theorem 1.5]), the phrase

“ $G$ -equivariant finite (resp. finite étale) Galois cover  $\tau : A \rightarrow X'$  (resp.  $\tau : A \rightarrow X$ )”

should be read as:

“equivariant finite (resp. finite étale) Galois cover  $\tau : (A, \tilde{G}) \rightarrow (X', G)$  (resp.  $\tau : (A, \tilde{G}) \rightarrow (X, G)$ ) with  $G \cong \tilde{G}/\text{Gal}(A/X')$  (resp.  $G \cong \tilde{G}/\text{Gal}(A/X)$ )” .

Such lifting from  $G$  on  $X'$  or  $X$  to  $\tilde{G}$  on  $A$  is due to [1, §3] applied to étale-in-codimension-one covers (cf. also [20, Proposition 3.5]). The proofs of [28, Theorem 1.5 and Corollaries 1.2 and 1.6] are not affected, while in the proof of [28, Theorem 1.1], one just replaces all  $G|_A$  by  $\tilde{G}$ , then the arguments go through. One also notes that  $N(\tilde{G})$  is the pullback of  $N(G)$  via the quotient map  $\gamma : \tilde{G} \rightarrow G$  (cf. [26, Remark 2.1(11)] or [19, Appendix A, Lemma A.8]) and hence  $\gamma$  induces an isomorphism  $\tilde{G}/N(\tilde{G}) \cong G/N(G)$ .

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